Existence of strong solutions for the compressible Ericksen-Leslie model

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Abstract: In this paper, we prove the existence and uniqueness of local strong solutions of the hydrodynamics of nematic liquid crystals system under the initial data satisfying a natural compatibility condition. Also the global strong solutions of the system with small initial data are obtained.

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1 Introduction

It is well known liquid crystals are states of matter which are capable of flow and in which the molecular arrangement gives rise to a preferred direction. By Ericksen-Leslie theory, the compressible nematic liquid crystals reads the following system:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \bigtriangledown p = \mu \bigtriangleup u - \lambda \operatorname{div}(\bigtriangledown d \odot \bigtriangledown d - \frac{1}{2}(|\bigtriangledown d|^2 + F(d))I), \tag{1.2}$$

$$d_t + u \cdot \bigtriangledown d = \nu(\bigtriangleup d - f(d)), \tag{1.3}$$

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in $\Omega \times (0,T)$, for a bounded smooth domain $\Omega \subset \mathbb{R}^3$. $\rho(t,x)$ is density, $u(t,x)$ the velocity field, $d(t,x)$ orientation parameter of the liquid crystal and $p(\rho)$ pressure with

$$p = p(\cdot) \in C^1[0,\infty), \qquad p(0) = 0. \tag{1.4}$$

The viscosity coefficients μ, λ, ν are positive constants. The unusual term $\bigtriangledown d \odot \bigtriangledown d$ denotes the 3×3 matrix whose (i,j)-th element is given by $\sum_{k=1}^{3} \partial_{x_i} d_k \partial_{x_j} d_k$. I is the unite matrix.

$$f(d) = \frac{1}{\sigma^2}(|d|^2 - 1)d \qquad \text{and} \qquad F(d) = \frac{1}{4\sigma^2}(|d|^2 - 1)^2.$$

We are interested in the initial data

$$\rho(0,x) = \rho_0 \geq 0, \quad u(0,x) = u_0, \quad d(0,x) = d_0(x), \quad \forall x \in \Omega, \tag{1.5}$$

and the boundary condition

$$u(t,x) = 0, \quad d(t,x) = d_0(x), \quad |d_0(x)| = 1, \quad \forall(t,x) \in (0,T) \times \partial\Omega. \tag{1.6}$$

The first author [1] firstly considers the model (1.1)-(1.4) and gives a global existence of finite energy weak solutions by using Lions's technique (see [2,3]). The incompressible model of a similar simplified Ericksen-Leslie model has been studied in the papers [4–8]. Recently, for the incompressible Ericksen-Leslie model, Wen-Ding [9] have gave the proof of local strong solutions in two and three dimensions, and Lin-Lin-Wang [10] have established the existence of global (in time) weak solutions on a bounded smooth domain in two dimensions .

Throughout this paper, we use the following simplified notations

$$L^q = L^q(\Omega), \quad W^{k,q} = W^{k,q}(\Omega), \quad H^k = W^{k,2}(\Omega), \quad H_0^1 = W_0^{1,2}(\Omega). \tag{1.7}$$

In order to obtain strong solutions, we need (ρ_0, u_0, d_0) satisfies the regularity

$$\rho_0 \in W^{1,6}, \qquad u_0 \in H_0^1 \cap H^2, \qquad d_0 \in H^3, \tag{1.8}$$

and a compatibility condition

$$\mu \triangle u_0 - \lambda \text{div}(\bigtriangledown d_0 \odot \bigtriangledown d_0 - \frac{1}{2}(|\bigtriangledown d_0|^2 + F(d_0))I) - \nabla p_0 = \rho_0^{\frac{1}{2}} g, \forall x \in \Omega, \qquad (1.9)$$

for some $g \in L^2$.

The compatibility condition is a natural request to insure $\|\sqrt{\rho} u_t(0)\|_{L^2}$ bounded, and a compensation to the lack of a positive lower bound of ρ_0. It is firstly introduced by Salvi, Strǎskraba [11] and the authors of [12] independently.

Our main result is the following theorem:

Theorem 1. **Part I: (Local existence)** Under the assumptions of the regularity condition (1.8) and the compatibility condition (1.9), there exists a small $T^* \in (0, T)$ such that the system (1.1)-(1.6) has a unique local strong solution (ρ, u, d) satisfying

$$
\begin{aligned}
&\rho \in C([0, T^*]; W^{1,6}), && \rho_t \in C([0, T^*]; L^6), \\
&u \in C([0, T^*]; H_0^1 \cap H^2) \cap L^2(0, T^*; W^{2,6}), && d \in C([0, T^*]; H^3), \\
&u_t \in L^2(0, T^*; H_0^1), && d_t \in C([0, T^*]; H_0^1) \cap L^2(0, T; H^2), \\
&\sqrt{\rho} u_t \in C([0, T^*]; L^2), && d_{tt} \in L^2(0, T^*; L^2).
\end{aligned}
\qquad (1.10)
$$

Part II: (Continuity of initial data) Suppose that $(\widetilde{\rho}, \widetilde{u}, \widetilde{d})$ is another solution of (1.1)-(1.4) with the following initial boundary conditions

$$
\begin{cases}
\widetilde{\rho}(0, x) = \widetilde{\rho}_0 \geq 0, & \widetilde{u}(0, x) = \widetilde{u}_0, & \widetilde{d}(0, x) = \widetilde{d}_0(x), & \forall x \in \Omega, \\
\widetilde{u}(t, x) = 0, & \widetilde{d}(t, x) = d_0(x), & \forall (t, x) \in (0, T) \times \partial \Omega,
\end{cases}
$$

then for any $t \in (0, T^*]$, the following quantities tend to zero when $(\widetilde{\rho}_0, \widetilde{u}_0, \widetilde{d}_0) \to (\rho_0, u_0, d_0)$ in $W^{1,6} \times H^2 \times H^3$:

$$\|d - \widetilde{d}\|_{H^2}(t), \ \|d - \widetilde{d}\|_{L^2(0,T^*;H^3)}, \ \|d_t - \widetilde{d}_t\|_{L^2}(t), \ \|d_t - \widetilde{d}_t\|_{L^2(0,T^*;H^1)},$$

$$\|u - \widetilde{u}\|_{H^1}(t), \ \|u - \widetilde{u}\|_{L^2(0,T^*;H^2)}, \ \|\sqrt{\rho}(u_t - \widetilde{u}_t)\|_{L^2(0,T^*;L^2)}, \ \|\rho - \widetilde{\rho}\|_{L^6}(t).$$

In general, as Navier-Stokes equations we have not the global existence with large initial data. However, a blow-up criterion is obtained in our article [13] and for small initial data, we have the following global existence.

Theorem 2. (**Global existence with small initial data**) Let α be a nonnegative constant and m a constant unite vector in $\mathbb{R}^3$. Then there exists a positive constant θ_0 small such that if the initial data satisfies

$$\max\{\|\rho_0 - \alpha\|_{W^{1,6}}, \|u_0\|_{H^2}, \|d_0 - m\|_{H^3}, \|g\|_{L^2}^2\} \leq \theta, \tag{1.11}$$

for all $\theta \in (0, \theta_0]$, then the system(1.1)-(1.6) has a unique global strong solution.

Remark 1. In this paper, we only consider the case $\alpha = 0$, because the other case $(\alpha > 0)$ can be induced to the problem with a positive initial data of density.

The methods to prove the Theorem 1 and Theorem 2 are the successive approximation in Sobolev spaces (see [14, 15]). We generalize this method to two variables, which is based on the following careful observation on coupling terms of d and u:

(1) $\|d\|_{H^2}^2$ can be deduced by $\int_0^t \|u\|_{H^2}^2 \mathrm{d}\tau$ and $\|u\|_{H^1}^2$;

(2) $\|u\|_{H^2}^2$ can be derived by $\int_0^t \|d\|_{H^2}^2 \mathrm{d}\tau$.

Since the initial density has vacuum, the movement equation (1.2) becomes a degenerate parabolic-elliptic couples system. To overcome this difficulty, as usual, the technique is to approximate the nonnegative initial data of density by a positive initial data. For a special linear equations of the system (1.1)-(1.3), we prove not only the local existence of strong solutions with large initial data but also the global existence of strong solutions with small initial data. Employing energy law and higher energy inequalities, we can prove both the uniqueness and the continuous dependence on the initial data.

This paper is written as follows. In section 2, after establishing a linear problem of the nonlinear problem (1.1)-(1.3), we prove local existence of a strong solution to the special linear problem with a positive initial data of density. And we also establish some uniform a prior estimates, which imply the existence of a local strong solution to the linear problem when the initial data of density allows vacuum in section 2.3. In section 3, after constructing a sequence of approximate solutions, a strong solution of the nonlinear problem (1.1)-(1.3) is obtained. In section 4, the uniqueness and continuous

dependence on initial data are proved.

2   A linear problem

2.1   Linearization

At the beginning, we linearize the equations (1.1)-(1.3) as following:

$$\rho_t + \text{div}(\rho v) = 0, \tag{2.1}$$

$$d_t + v \cdot \bigtriangledown d = \nu(\triangle d - \frac{1}{\sigma^2}(n+m) \cdot (d-m)n), \tag{2.2}$$

$$\rho u_t + \rho v \cdot u + \bigtriangledown p(\rho) = \mu \triangle u - \lambda(\bigtriangledown d)^T(\triangle d - f(d)), \tag{2.3}$$

where m is a constant unite vector and (v, n) satisfies the following regularity

$$v \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; W^{2,6}), \qquad v_t \in L^2(0,T; H_0^1),$$
$$n \in C([0,T]; H^2) \cap L^2(0,T; H^3), \qquad n_t \in C([0,T]; H_0^1) \cap L^2(0,T; H^2). \tag{2.4}$$

Assume further that

$$\begin{cases} v(0,x) = u_0(x), & n(0,x) = d_0(x), & \forall x \in \Omega, \\ v(t,x) = 0, & n(t,x) = d_0(x), & \forall(t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{2.5}$$

2.2   The existence of approximate solutions

For each $\delta \in (0,1)$, let u_0^δ be the solution to the boundary value problem:

$$\begin{cases} \mu \triangle u_0^\delta - \lambda\text{div}(\bigtriangledown d_0 \odot \bigtriangledown d_0 - \frac{1}{2}(|\bigtriangledown d_0|^2 + F(d_0))I) - p(\rho_0^\delta) = (\rho_0^\delta)^{\frac{1}{2}}g, \\ u_0^\delta = 0, \forall x \in \partial\Omega, \end{cases} \tag{2.6}$$

where $\rho_0^\delta = \rho_0 + \delta$. Then $u_0^\delta \to u_0$ in $H_0^1 \cap H^2$ as $\delta \to 0$.

For the linear equations (2.1)-(2.3), we have the following theorem.

Theorem 2.1. *Assume ρ_0, u_0 and d_0 satisfy the regularity (1.8) and (v, n) satisfies the above conditions (2.4) and (2.5). Then there exists a unique strong solution (ρ, u, d)*

of the linear equations (2.1)-(2.3) with the initial data $(\rho_0^\delta, u_0^\delta, d_0)$ and the boundary condition (1.6) such that

$$
\begin{aligned}
&\rho \in C([0,T]; W^{1,6}), && \rho_t \in C([0,T]; L^6), \\
&u \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; W^{2,6}), && u_t \in C([0,T]; L^2) \cap L^2(0,T; H_0^1), \\
&d \in C([0,T]; H^2) \cap L^2(0,T; H^3), && d_t \in C([0,T]; H_0^1) \cap L^2(0,T; H^2), \\
&u_{tt} \in L^2(0,T; H^{-1}), && d_{tt} \in L^2(0,T; L^2).
\end{aligned}
\tag{2.7}
$$

We will prove the theorem by the following three lemmas. Firstly, suppose the constants c_0, c_1 and c_2 satisfying

$$
\begin{aligned}
c_0 \;>\;& 1 + \|\rho_0\|_{W^{1,6}} + \|d_0\|_{H^3} + \|u_0\|_{H^2} + \|g\|_{L^2}, \tag{2.8} \\
c_1 \;>\;& \sup_{0 \le t \le T} (\|v\|_{H_0^1} + \|n\|_{H^1} + \|n_t\|_{H_0^1}) + \int_0^T (\|\nabla v_t\|_{L^2}^2 + \|v\|_{W^{2,6}}^2 \\
& + \|\nabla^2 n_t\|_{L^2}^2 + \|n\|_{H^3}^2) \mathrm{d}t, \tag{2.9} \\
c_2 \;>\;& \sup_{0 \le t \le T} (\|\nabla^2 v\|_{L^2} + \|\nabla^2 n\|_{L^2}), \tag{2.10} \\
c_2 \;>\;& c_1 > c_0 > 1. \tag{2.11}
\end{aligned}
$$

Lemma 2.1. *Assume ρ_0 and v satisfy the regularities (1.8) and (2.4) respectively. Then the problem (2.1) and (1.5) has a global unique strong solution such that*

$$
\rho \in C([0,T]; W^{1,6}), \quad \rho_t \in C([0,T]; L^6). \tag{2.12}
$$

Moreover, if v, n satisfy (2.8)-(2.11), then there is a $T_1 = \min\{c_1^{-1}, T\}$ such that $\forall t \in [0, T_1]$,

$$
\|\rho\|_{W^{1,6}}(t) \le Cc_0, \quad \|\rho_t\|_{L^6}(t) \le Cc_0c_2. \tag{2.13}
$$

In particular,

$$
\|p\|_{L^6}(t) \le CM(c_0), \quad \|\nabla p\|_{L^6}(t) \le CM(c_0)c_0, \quad \|p_t\|_{L^6}(t) \le CM(c_0)c_0c_2, \tag{2.14}
$$

where the constant $M(c_0)$ is defined by (2.18).

Remark 2. The above lemma is also true to the problem (2.1) with the initial data ρ_0^δ and the estimates 2.13-2.14 also hold for all small δ.

Proof. Now let's start to prove Lemma 2.1. The existence is obvious, based on the classical method of characteristics (see [15]). So

$$\rho(t, x) = \rho_0(y(0, t, x)) \exp(- \int_0^t \operatorname{div} v(\tau, y(\tau, t, x)) \mathrm{d}\tau), \tag{2.15}$$

where $y(\tau, t, x)(\in C([0, T] \times [0, T] \times \overline{\Omega}))$ is a solution of the initial value problem:

$$\begin{cases} \frac{\partial}{\partial \tau} y(\tau, t, x) = v(\tau, y(\tau, t, x)), & 0 \leq \tau \leq T, \\ y(t, t, x) = x, & 0 \leq t \leq T, \ x \in \overline{\Omega}. \end{cases}$$

Moreover, we have

$$\|\rho\|_{W^{1,6}}(t) \leq \|\rho_0\|_{W^{1,6}} \exp(\int_0^t \|\nabla v\|_{W^{1,6}}(\tau) \mathrm{d}\tau), \qquad \forall t \in [0, T]. \tag{2.16}$$

So that for all $t \in [0, T_1]$,

$$\begin{aligned} \|\rho\|_{W^{1,6}}(t) &\leq \|\rho_0\|_{W^{1,6}} \exp(\int_0^t \|\nabla v\|_{W^{1,6}}(\tau) \mathrm{d}\tau) \\ &\leq c_0 \exp(t^{\frac{1}{2}} (\int_0^t \|\nabla v\|_{W^{1,6}}^2(\tau) \mathrm{d}\tau)^{\frac{1}{2}}) \\ &\leq e c_0. \end{aligned} \tag{2.17}$$

Because of $\|\rho\|_{L^\infty} \leq \widetilde{C} \|\rho\|_{W^{1,6}}$, set

$$M(c_0) = \sup_{0 \leq \cdot \leq \widetilde{C} e c_0} (1 + p(\cdot) + p'(\cdot)). \tag{2.18}$$

We can obtain for all $t \in [0, T_1]$,

$$\begin{aligned} \|\rho_t\|_{L^6}(t) &= \| - v \cdot \nabla \rho - \rho \operatorname{div} v \|_{L^6} \\ &\leq \|v\|_{L^\infty} \|\nabla \rho\|_{L^6} + \|\rho\|_{L^\infty} \|\nabla v\|_{L^6} \\ &\leq \|v\|_{H^2} \|\rho\|_{W^{1,6}} + \|\rho\|_{W^{1,6}} \|v\|_{H^2} \\ &\leq C c_0 c_2. \end{aligned}$$

Similarly, for all $t \in [0, T_1]$,

$$\|p\|_{L^6}(t) \le CM(c_0), \qquad \|\triangledown p\|_{L^6}(t) \le CM(c_0)c_0,$$

$$\|p_t\|_{L^6}(t) \le CM(c_0)c_2c_0.$$

$\square$

Lemma 2.2. *Under the hypotheses of Theorem 2.1, then the equation (2.2) with the initial boundary conditions (1.5)-(1.6) has a global unique strong solution d satisfying (2.7).*

Moreover, if v, n satisfy (2.8)-(2.11), then there is a $T_3 = \min\{c_2^{-22}, T\}$ such that

$$\sup_{0 \le t \le T_3} (\|d\|_{H^1} + c_1^{-2}\|\triangledown^2 d\|_{L^2} + c_0^{\frac{3}{2}}\|d_t\|_{H^1} + c_2^{-2}c_1^{-1}\|\triangledown d\|_{H^2})$$

$$+ \int_0^{T_3} \|d_t\|_{H^2}^2 + c_0^3\|d\|_{H^3}^2 \mathrm{d}t \le Cc_0^3. \tag{2.19}$$

Proof. By the classic Galerkin method to the linear parabolic equation (2.2) with (1.5)-(1.6), the existence and regularity of d described in (2.7) can be obtained.

Differentiating (2.2) with respect to time, multiplying by d_t and then integrating over Ω, we can deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |d_t|^2 \mathrm{d}x + \nu \int_\Omega |\triangledown d_t|^2 \mathrm{d}x$$

$$\le C(\|v_t\|_{L^6}\|\triangledown d\|_{L^2}\|d_t\|_{L^3} + \|v\|_{L^\infty}\|\triangledown d_t\|_{L^2}\|d_t\|_{L^2} + \|n\|_{L^3}\|n_t\|_{L^6}\|d - m\|_{L^3}\|d_t\|_{L^6}$$

$$+ \|n + m\|_{L^3}\|n_t\|_{L^6}\|d - m\|_{L^3}\|d_t\|_{L^6} + \|n\|_{L^6}\|n + m\|_{L^6}\|d_t\|_{L^3}^2)$$

$$= \sum_{i=1}^5 I_i. \tag{2.20}$$

Here

$$I_1 \quad \le C\eta\|v_t\|_{H^1}^2\|\triangledown d\|_{L^2}^2 + C\eta^{-1}\|d_t\|_{L^2}^2 + \frac{\nu}{5}\|\triangledown d_t\|_{L^2}^2,$$

$$I_2 \quad \le C\|v\|_{H^2}^2\|d_t\|_{L^2}^2 + \frac{\nu}{5}\|\triangledown d_t\|_{L^2}^2,$$

$$I_3 + I_4 \quad \le C((\|n\|_{L^2}^2 + 1)(\|n\|_{H^1}^2 + 1)\|n_t\|_{L^6}^4\|d - m\|_{L^2}^2 + \|\triangledown d\|_{L^2}^2 + \|d - m\|_{L^2}^2)$$

$$+ \frac{\nu}{5}\|\triangledown d_t\|_{L^2}^2,$$

$$I_5 \quad \le C\|n\|_{H^1}^2\|n + m\|_{H^1}^2\|d_t\|_{L^2}^2 + \frac{\nu}{5}\|\triangledown d_t\|_{L^2}^2,$$

where the small positive constant η will be fixed later.

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\bigtriangledown d|^2\mathrm{d}x = 2\int_\Omega \bigtriangledown d : \bigtriangledown d_t\mathrm{d}x \leq \int_\Omega |\bigtriangledown d|^2\mathrm{d}x + \int_\Omega |\bigtriangledown d_t|^2\mathrm{d}x, \tag{2.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |d - m|^2\mathrm{d}x \leq \int_\Omega |d - m|^2\mathrm{d}x + \int_\Omega |d_t|^2\mathrm{d}x, \tag{2.22}$$

combining (2.20), (2.21) and (2.22), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |d_t|^2 + |\bigtriangledown d|^2 + |d - m|^2\mathrm{d}x + \int_\Omega |\bigtriangledown d_t|^2\mathrm{d}x$$

$$\leq C(\|\bigtriangledown d\|_{L^2}^2 + \|d - m\|_{L^2}^2 + \|d_t\|_{L^2}^2) \cdot (\eta^{-1} + \|v\|_{H^2}^2 + \|n\|_{H^1}^2\|n + m\|_{H^1}^2$$

$$+ \eta\|v_t\|_{H^1}^2 + (\|n\|_{L^2}^2 + 1)(\|n\|_{H^1}^2 + 1)\|\bigtriangledown n_t\|_{L^2}^4 + 1). \tag{2.23}$$

From the equation (2.2), we can deduce

$$\|d_t\|_{L^2}(0) \leq C(\|\triangle d_0\|_{L^2} + \|u_0\|_{H^2}\|\bigtriangledown d_0\|_{L^2} + \|d_0 + m\|_{L^2}\|d_0 - m\|_{L^2}\|d_0\|_{L^2})$$

$$\leq Cc_0^3. \tag{2.24}$$

Hence, by Gronwall's inequality, we can deduce from (2.23)

$$\int_\Omega |d_t|^2 + |\bigtriangledown d|^2 + |d - m|^2\mathrm{d}x + \int_0^t\!\!\int_\Omega |\bigtriangledown d_t|^2\mathrm{d}x\mathrm{d}\tau$$

$$\leq Cc_0^6\exp(C\int_0^t \eta^{-1} + \|v\|_{H^2}^2 + \|n\|_{H^1}^2 + \|n + m\|_{H^1}^2 + \eta\|v_t\|_{H^1}^2$$

$$+ (\|n\|_{L^2}^2 + 1)(\|n\|_{H^1}^2 + 1)\|\bigtriangledown n_t\|_{L^2}^4\mathrm{d}\tau). \tag{2.25}$$

Taking $\eta = c_1^{-1}$ and using the assumption (2.8)-(2.11), we obtain

$$\sup_{0\leq t\leq T_2}\int_\Omega |d_t|^2 + |\bigtriangledown d|^2 + |d - m|^2\mathrm{d}x + \int_0^{T_2}\!\!\int_\Omega |\bigtriangledown d_t|^2\mathrm{d}x\mathrm{d}\tau \leq Cc_0^6, \tag{2.26}$$

where $T_2 = \min\{c_2^{-8}, T_1\}$.

From the equation (2.2) and using the elliptic estimates, we get

$$\|d - m\|_{H^2} \leq C(\|d_0 - m\|_{H^2} + \|d_t\|_{L^2} + \|v \cdot \bigtriangledown d\|_{L^2} + \|(n + m) \cdot (d - m)n\|_{L^2})$$

$$\leq C(c_0^3 + \|v\|_{H^1}\|\bigtriangledown d\|_{L^2}^{\frac{1}{2}}\|d - m\|_{H^2}^{\frac{1}{2}} + \|n + m\|_{L^6}\|d - m\|_{L^6}\|n\|_{L^6})$$

$$\leq C(c_0^3 + \|v\|_{H^1}^2\|\bigtriangledown d\|_{L^2} + \|n + m\|_{H^1}\|d - m\|_{H^1}\|n\|_{H^1}) + \frac{1}{2}\|d\|_{H^2}.$$

So

$$\| \nabla^2 d \|_{L^2} \leq C(c_0^3 + c_1^2 c_0^3 + c_1^2 c_0^3) \leq C c_1^2 c_0^3. \tag{2.27}$$

Applying the operator ∇ to the linear equation (2.2), we get

$$\nu \triangle (\nabla d) = \nabla d_t + \nabla(v \cdot \nabla d) + \frac{\nu}{\sigma^2} \nabla ((n+m) \cdot (d-m)n). \tag{2.28}$$

By the elliptic estimates, we can estimate the term $\| \nabla (d-m) \|_{H^2}$ as follows

$$
\begin{aligned}
\| \nabla (d-m) \|_{H^2} \leq & C(\| \nabla d_t \|_{L^2} + \| \nabla (v \cdot \nabla d) \|_{L^2} + \| \frac{\nu}{\sigma^2} \nabla [(n+m) \cdot (d-m)n] \|_{L^2} \\
& + \| d_0 - m \|_{H^3}) \\
\leq & C(\| \nabla d_t \|_{L^2} + \| \nabla v \|_{H^1} \| \nabla d \|_{L^2}^{\frac{1}{2}} \| \nabla d \|_{H^1}^{\frac{1}{2}} + \| v \|_{H^2} \| \nabla^2 d \|_{L^2} \\
& + \| \nabla n \|_{H^1} \| n \|_{H^1} \| d-m \|_{H^1} + \| n \|_{H^2}^2 \| \nabla d \|_{L^2} + \| d_0 - m \|_{H^3}) \\
\leq & C(\| \nabla d_t \|_{L^2} + c_2^2 c_1 c_0^3), \tag{2.29}
\end{aligned}
$$

where we use the assumption (2.8)-(2.11) and (2.26)-(2.27).

Differentiating (2.2) with respect to time and taking inner product with $\triangle d_t$, then we can derive

$$
\begin{aligned}
& \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} | \nabla d_t |^2 \mathrm{d}x + \frac{\nu}{C} \| d_t \|_{H^2}^2 \\
\leq & \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} | \nabla d_t |^2 \mathrm{d}x + \nu \int_{\Omega} | \triangle d_t |^2 \mathrm{d}x \\
= & \int_{\Omega} (v_t \cdot \nabla d) \triangle d_t \mathrm{d}x + \int_{\Omega} (v \cdot \nabla d_t) \triangle d_t \mathrm{d}x + \frac{\nu}{\sigma^2} \int_{\Omega} (n_t \cdot (d-m))n \triangle d_t \mathrm{d}x \\
& + \frac{\nu}{\sigma^2} \int_{\Omega} (n \cdot d_t)n \triangle d_t \mathrm{d}x + \frac{\nu}{\sigma^2} \int_{\Omega} ((n+m) \cdot (d-m))n_t \triangle d_t \mathrm{d}x \\
= & \sum_{j=1}^{5} J_j, \tag{2.30}
\end{aligned}
$$

where we use the elliptic estimate $\| d_t \|_{H^2}^2 \leq C \| \triangle d_t \|_{L^2}^2$.

Here

$$
\begin{aligned}
|J_1| &= \left| \int_\Omega (v_t \cdot \nabla d) \, \triangle d_t \mathrm{d}x \right| \\
&\leq \quad \|\nabla v_t\|_{L^2} \|\nabla d\|_{L^6} \|\nabla d_t\|_{L^3} + \|v_t\|_{L^3} \|\nabla^2 d\|_{L^6} \|\nabla d_t\|_{L^2} \\
&\leq \quad \eta \|\nabla v_t\|_{L^2}^2 + \eta^{-1} \|\nabla d\|_{H^1}^2 \|\nabla d_t\|_{L^2} \|\nabla d_t\|_{H^1} + \eta \|v_t\|_{L^3}^2 \|\nabla d_t\|_{L^2}^2 \\
&\qquad + \eta^{-1} \|\nabla^2 d\|_{L^6}^2 \\
&\leq \quad \eta \|\nabla v_t\|_{L^2}^2 + C\eta^{-2} \varepsilon^{-1} c_1^8 c_0^{12} \|\nabla d_t\|_{L^2}^2 + \varepsilon \|d_t\|_{H^2}^2 + \eta \|v_t\|_{H^1}^2 \|\nabla d_t\|_{L^2}^2 \\
&\qquad + C\eta^{-1} (\|\nabla d_t\|_{L^2}^2 + c_2^4 c_1^2 c_0^6),
\end{aligned}
$$

$$
\begin{aligned}
|J_2| &= \left| \int_\Omega (v \cdot \nabla d_t) \, \triangle d_t \mathrm{d}x \right| \leq C\varepsilon^{-1} c_2^2 \|\nabla d_t\|_{L^2}^2 + \varepsilon \|\triangle d_t\|_{L^2}^2, \\
|J_4| &\leq \quad C\varepsilon^{-1} \|n\|_{L^\infty}^4 \|d_t\|_{L^2}^2 + \varepsilon \|\triangle d_t\|_{L^2}^2 \leq C\varepsilon^{-1} c_2^4 c_0^6 + \varepsilon \|\triangle d_t\|_{L^2}^2, \\
|J_3| + |J_5| &\leq \quad C\varepsilon^{-1} (\|n\|_{L^\infty}^2 + 1) \|n_t\|_{L^2}^2 \|d - m\|_{L^\infty}^2 + \varepsilon \|\triangle d_t\|_{L^2}^2 \\
&\leq \quad C\varepsilon^{-1} c_2^2 c_1^6 c_0^6 + \varepsilon \|\triangle d_t\|_{L^2}^2,
\end{aligned}
$$

where the small positive constants ε and η will be fixed later.

Substituting the above $J_1 - J_5$ into (2.30) and taking ε small enough, we get

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla d_t|^2 \mathrm{d}x + \|d_t\|_{H^2}^2 \leq A_\eta(t) \|\nabla d_t\|_{L^2}^2 + B_\eta(t), \tag{2.31}
$$

where

$$
\begin{aligned}
A_\eta(t) &= \quad C(\eta^{-2} c_1^8 c_0^{12} + \eta \|v_t\|_{H^1}^2 + \eta^{-1} + c_2^2), \\
B_\eta(t) &= \quad C(\eta \|v_t\|_{H^1}^2 + \eta^{-1} c_2^4 c_1^2 c_0^8 + \eta^{-1} c_2^2 c_1^6 c_0^3 + c_2^4 c_0^6).
\end{aligned}
$$

In view of the inequalities (2.26)-(2.27) and taking $\eta = c_2^{-1}$, we have for all $t \in [0, T_2]$,

$$
\int_0^t A_\eta(s) \mathrm{d}s \leq C + C c_2^2 c_1^8 c_0^{12} t, \quad \int_0^t B_\eta(s) \mathrm{d}s \leq C + C c_2^5 c_1^4 c_0^6 t. \tag{2.32}
$$

From the equation (2.28), we can estimate the term $\|\nabla d_t(\tau)\|_{L^2}$ at time 0

$$
\begin{aligned}
\|\nabla d_t(0)\|_{L^2} &\leq \quad C(\|d_0 - m\|_{H^3} + \|\nabla u_0\|_{L^3} \|\nabla^2 d_0\|_{L^6} + \|u_0\|_{L^3} \|\nabla^2 d_0\|_{L^6} \\
&\qquad + \|\nabla d_0\|_{L^6} \|d_0 - m\|_{L^6} \|d_0\|_{L^6} + \|d_0\|_{L^6}^2 \|\nabla d_0\|_{L^6}) \\
&\leq \quad C c_0^3. \tag{2.33}
\end{aligned}
$$

Applying Gronwall's inequality to (2.31) and using (2.32)-(2.33), we can obtain

$$\sup_{0 \leq t \leq T_3} \int_\Omega | \bigtriangledown d_t|^2 \mathrm{d}x + \int_0^{T_3} \|d_t\|_{H^2}^2 \mathrm{d}t \leq C(c_0^3 + c_2^5 c_1^4 c_0^6 T_3) \exp\left(C + C c_2^2 c_1^8 c_0^{12} T_3\right)$$

$$\leq C c_0^3. \tag{2.34}$$

From (2.29) and (2.34), we have

$$\| \bigtriangledown d\|_{H^2}(t) \leq C(c_0^{\frac{3}{2}} + c_2^2 c_1 c_0^3), \ \forall t \in [0, T_3]. \tag{2.35}$$

Hence

$$\int_0^t \|d\|_{H^3}^2 \mathrm{d}\tau \leq C, \qquad \forall t \in [0, T_3]. \tag{2.36}$$

Since (2.26), (2.27), (2.34) and (2.36) can deduce the estimate (2.19), we complete the proof. $\square$

Lemma 2.3. *Under the hypotheses of Theorem 2.1, then there exists a global unique solution u of the equation (2.3) with the initial data u_0^δ and the boundary condition (1.6), and the solution u has the regularity in (2.7).*

Moreover, if v, n satisfy (2.8)-(2.11), then, for all small δ,

$$\sup_{0 \leq t \leq T_3} (M(c_0) c_0^{13} \|u\|_{H_0^1} + M(c_0) c_0^8 c_1^{-6} \| \bigtriangledown^2 u\|_{L^2} + M(c_0) c_0^{13} \|\sqrt{\rho} u_t\|_{L^2})$$
$$+ \int_0^{T_3} c_0^8 \| \bigtriangledown u_t\|_{L^2}^2 + \|u\|_{W^{2,6}}^2 \mathrm{d}t \leq C c_0^{18} M^2(c_0). \tag{2.37}$$

Proof. Because $\rho_0^\delta \geq \delta > 0$, it follows from the representation (2.15) that

$$\rho(t, x) \geq \delta \exp\left(- \int_0^t | \bigtriangledown v(\tau)|_{W^{1,6}} \mathrm{d}\tau\right) \geq \underline{\delta}, \qquad \forall(t, x) \in [0, T] \times \overline{\Omega}, \tag{2.38}$$

where $\underline{\delta}$ is a positive constant.

Thanks to (2.38), we change (2.3) into the following form:

$$u_t + v \cdot \bigtriangledown u + \frac{1}{\rho} \bigtriangledown p = \frac{\mu}{\rho} \bigtriangleup u - \frac{\lambda}{\rho} (\bigtriangledown d)^T (\bigtriangleup d - f(d)).$$

Applying the Galerkin method again to the above equation with the initial data u_0^δ and the boundary condition (1.6), we can deduce the existence and regularity of u described in (2.7).

Differentiating (2.3) with respect to time t, multiplying by u_t and then integrating over Ω, we can derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|u_t|^2\mathrm{d}x + \mu\int_\Omega |\bigtriangledown u_t|^2\mathrm{d}x$$

$$= \int_\Omega (-\bigtriangledown p_t - \rho_t v\cdot\bigtriangledown u - 2\rho v\cdot\bigtriangledown u_t - \rho v_t\cdot\bigtriangledown u))u_t\mathrm{d}x$$

$$-\lambda\int_\Omega(\bigtriangledown d_t)^T(\triangle d - f(d))u_t\mathrm{d}x - \lambda\int_\Omega(\bigtriangledown d)^T(\triangle d - f(d))_t u_t\mathrm{d}x$$

$$= \sum_{k=1}^{6}K_k. \tag{2.39}$$

Here

$$|K_1| \leq C\|p_t\|_{L^2}^2 + \frac{\mu}{8}\|\bigtriangledown u_t\|_{L^2}^2 \leq CM^2(c_0)c_0^2 c_2^2 + \frac{\mu}{8}\|\bigtriangledown u_t\|_{L^2}^2,$$

$$|K_2| \leq C\|\rho_t\|_{L^6}^2\|v\|_{L^6}^2\|\bigtriangledown u\|_{L^2}^2 + \frac{\mu}{8}\|\bigtriangledown u_t\|_{L^2}^2 \leq Cc_2^2 c_1^2 c_0^2\|\bigtriangledown u\|_{L^2}^2 + \frac{\mu}{8}\|\bigtriangledown u_t\|_{L^2}^2,$$

$$|K_3| \leq C\|\rho\|_{L^\infty}\|v\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{\mu}{8}\|\bigtriangledown u_t\|_{L^2}^2 \leq Cc_2^2 c_0\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{\mu}{8}\|\bigtriangledown u_t\|_{L^2}^2,$$

$$|K_4| \leq \eta\|\bigtriangledown v_t\|_{L^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + \eta^{-1}\|\rho\|_{L^\infty}\|\bigtriangledown u\|_{L^2}\|\bigtriangledown u\|_{H^1}$$

$$\leq \eta\|\bigtriangledown v_t\|_{L^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + \eta^{-2}c_0^2\|\bigtriangledown u\|_{L^2}^2 + \|\bigtriangledown u\|_{H^1}^2,$$

$$\begin{aligned}|K_5| \quad \leq \quad & C(\|\bigtriangledown^2 d\|_{L^3}\|\bigtriangledown d_t\|_{L^2}\|u_t\|_{L^6} + \|\bigtriangledown d_t\|_{L^2}\|d\|_{L^3}\|u_t\|_{L^6}\|d\|_{L^\infty}^2 \\ & + \|\bigtriangledown d_t\|_{L^2}\|d\|_{L^3}\|u_t\|_{L^6}) \\ \leq \quad & C\|\bigtriangledown^2 d\|_{L^2}\|\bigtriangledown^2 d\|_{L^6}\|\bigtriangledown d_t\|_{L^2}^2 + C\|\bigtriangledown d_t\|_{L^2}^2\|d\|_{L^2}\|d\|_{H^1}\|d\|_{H^2}^4 \\ & + C\|\bigtriangledown d_t\|_{L^2}^2\|d\|_{L^2}\|d\|_{H^1} + \frac{\mu}{8\lambda}\|\bigtriangledown u_t\|_{L^2}^2 \\ \leq \quad & Cc_2^2 c_1^6 c_0^{21} + \frac{\mu}{8\lambda}\|\bigtriangledown u_t\|_{L^2}^2,\end{aligned}$$

$$
\begin{aligned}
|K_6| \leq &C(\|\nabla^2 d\|_{L^3}\|\nabla d_t\|_{L^2}\|u_t\|_{L^6} + \|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2}\|\nabla u_t\|_{L^2} \\
&+ \|\nabla d\|_{L^2}\|d_t\|_{L^3}\|u_t\|_{L^6} + \|\nabla d\|_{L^3}\|d_t\|_{L^6}\|d\|_{L^6}^2\|u_t\|_{L^6}) \\
\leq &C(\|\nabla^2 d\|_{L^2}\|\nabla^2 d\|_{L^6}\|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{W^{1,6}}^2\|\nabla d_t\|_{L^2}^2 \\
&+ \|d_t\|_{H^1}\|d_t\|_{L^2}\|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}\|\nabla d\|_{H^1}\|\nabla d_t\|_{L^2}^2\|d\|_{H^1}^4) \\
&+ \frac{\mu}{8\lambda}\|\nabla u_t\|_{L^2}^2 \\
\leq &Cc_2^4 c_1^2 c_0^{17} + \frac{\mu}{8\lambda}\|\nabla u_t\|_{L^2}^2.
\end{aligned}
$$

On the other hand

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 = \int_\Omega \nabla u : \nabla u_t \mathrm{d}x \leq C\|\nabla u\|_{L^2}^2 + \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2. \tag{2.40}
$$

Substituting $|K_1| - |K_6|$ into (2.39) and combing with (2.40), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\rho|u_t|^2 + |\nabla u|^2)\mathrm{d}x + \int_\Omega |\nabla u_t|^2\mathrm{d}x \\
&\leq \quad \mathscr{A}_\eta(t)(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) + \mathscr{B}(t) + C\|\nabla u\|_{H^1}^2, \tag{2.41}
\end{aligned}
$$

where

$$
\mathscr{A}_\eta(t) = C(c_2^2 c_1^2 c_0^2 + \eta\|\nabla v_t\|_{L^2}^2 + \eta^{-2}c_0^2), \quad \mathscr{B}(t) = Cc_2^4 c_1^4 c_0^{21} M^2(c_0).
$$

Taking $\eta = c_2^{-1}$ and using the estimates (2.13)-(2.14) and (2.19), we have $\forall t \in [0, T_3]$,

$$
\int_0^t \mathscr{A}_\eta(s)\mathrm{d}s \leq C + Cc_2^4 t, \quad \int_0^t \mathscr{B}(s)\mathrm{d}s \leq Cc_2^4 c_1^4 c_0^{21} M^2(c_0)t + C. \tag{2.42}
$$

Multiplying (2.3) by u_t, then integrating it over Ω and using the Young's inequality, we can obtain

$$
\begin{aligned}
\int_\Omega \rho|u_t|^2\mathrm{d}x(\tau) \leq \quad & C\int_\Omega \rho|v|^2|\nabla u|^2 + |\rho^{-\frac{1}{2}}(\mu\triangle u - \lambda\mathrm{div}(\nabla d \odot \nabla d \\
&- \frac{1}{2}(|\nabla d|^2 + F(d))I) - \nabla p)|^2\mathrm{d}x(\tau).
\end{aligned}
$$

Hence

$$
\limsup_{\tau\to 0+}\int \rho|u_t|^2\mathrm{d}x(\tau) \leq C(c_0^5 + \|g\|_{L^2}^2) \leq Cc_0^5. \tag{2.43}
$$

Integrating (2.41) with respect to time (τ, t) and letting $\tau \to 0+$, thanks to (2.43) and Gronwall's inequality and the inequality (2.42), we have for all $t \in [0, T_3]$,

$$\int_\Omega (\rho|u_t|^2 + |\bigtriangledown u|^2)\mathrm{d}x(t) + \int_0^t\!\!\int_\Omega |\bigtriangledown u_t|^2\mathrm{d}x\mathrm{d}\tau$$

$$\leq Cc_0^7 M^2(c_0) + C\int_0^t \|\bigtriangledown u\|_{H^1}^2 \mathrm{d}\tau. \tag{2.44}$$

Using the elliptic regularity result to the linear movement equation (2.3), we can estimate the term $\|\bigtriangledown^2 u\|_{L^2}$ as follows

$$\|\bigtriangledown u\|_{H^1} \leq C(\|\rho u_t\|_{L^2} + \|\rho v \cdot \bigtriangledown u\|_{L^2} + \|\bigtriangledown p\|_{L^2} + \|\bigtriangledown u\|_{L^2}$$

$$+\|(\bigtriangledown d)^T(\triangle d - f(d))\|_{L^2}). \tag{2.45}$$

From the assumption (2.8)-(2.11) and the estimates (2.13), (2.14) and (2.19), we can derive

$$\|(\bigtriangledown d)^T \triangle d\|_{L^2} \leq C\|(\bigtriangledown d)^T(d_t + v \cdot \bigtriangledown d)\|_{L^2} + C\|(\bigtriangledown d)^T[(n+m)\cdot(d-m)]n\|_{L^2}$$

$$\leq C(\|\bigtriangledown d\|_{L^2}^{\frac{1}{2}}\|\bigtriangledown d\|_{H^1}^{\frac{1}{2}}\|\bigtriangledown d_t\|_{L^2} + \|\bigtriangledown d\|_{H^1}\|v\|_{H^1}\|\bigtriangledown d\|_{H^1}$$

$$+\|d\|_{H^2}\|n+m\|_{L^6}\|d-m\|_{H^2}\|n\|_{L^6})$$

$$\leq Cc_1^6 c_0^6, \tag{2.46}$$

$$\|(\bigtriangledown d)^T f(d)\|_{L^2} \leq C\|\bigtriangledown d\|_{H^1}\|d+m\|_{L^6}\|d-m\|_{L^6}\|d\|_{H^2} \leq Cc_1^4 c_0^{12} \tag{2.47}$$

and

$$\|\rho u_t\|_{L^2} + \|\rho v \cdot \bigtriangledown u\|_{L^2} + \|\bigtriangledown p\|_{L^2} + \|\bigtriangledown u\|_{L^2}$$

$$\leq \|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + \|\rho\|_{L^\infty}\|v\|_{L^6}\|\bigtriangledown u\|_{L^3} + CM(c_0)c_0 + \|\bigtriangledown u\|_{L^2}$$

$$\leq Cc_0^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2} + Cc_0 c_1\|\bigtriangledown u\|_{L^2}^{\frac{1}{2}}\|\bigtriangledown u\|_{H^1}^{\frac{1}{2}} + CM(c_0)c_0 + \|\bigtriangledown u\|_{L^2}$$

$$\leq CM(c_0)c_0 + Cc_1^4(\|\bigtriangledown u\|_{L^2} + \|\sqrt{\rho}u_t\|_{L^2}) + \frac{1}{2}\|\bigtriangledown u\|_{H^1}. \tag{2.48}$$

Taking (2.46)-(2.48) into (2.45), we can deduce

$$\|\bigtriangledown u\|_{H^1} \leq Cc_1^6 c_0^{10} + CM(c_0)c_0 + Cc_1^4(\|\bigtriangledown u\|_{L^2} + \|\sqrt{\rho}u_t\|_{L^2}). \tag{2.49}$$

Taking (2.49) into (2.44) and using Gronwall's inequality, we can derive for all $t \in [0, T_3]$

$$\int_\Omega (\rho |u_t|^2 + |\bigtriangledown u|^2) \mathrm{d}x + \int_0^t \!\!\!\int_\Omega |\bigtriangledown u_t|^2 \mathrm{d}x \mathrm{d}\tau \leq C c_0^{10} M^2(c_0). \tag{2.50}$$

So substituting (2.50) into (2.49), we get

$$\| \bigtriangledown u \|_{H^1} \leq C c_1^6 c_0^{10} M(c_0). \tag{2.51}$$

Using the elliptic regularity result to the linear movement equation(2.3), the term $\| \bigtriangledown^2 u \|_{L^6}$ can be controlled as follows

$$
\begin{aligned}
\| \bigtriangledown^2 u \|_{L^6} &\leq C(\|\rho u_t\|_{L^6} + \|\rho v \cdot \bigtriangledown u\|_{L^6} + \| \bigtriangledown p \|_{L^6} + \|(\bigtriangledown d)^T (\triangle d - f(d))\|_{L^6} \\
&\quad + \| \bigtriangledown u \|_{L^6}).
\end{aligned}
\tag{2.52}
$$

It follows from the assumption (2.8)-(2.11) and the estimates (2.13), (2.14) and (2.51) that

$$
\begin{aligned}
&\|\rho u_t\|_{L^6} + \|\rho v \cdot \bigtriangledown u\|_{L^6} + \| \bigtriangledown p \|_{L^6} + \| \bigtriangledown u \|_{L^6} \\
\leq\ & \|\rho\|_{L^\infty} \|u_t\|_{L^6} + \|\rho\|_\infty \|v\|_{L^\infty} \| \bigtriangledown u \|_{L^6} + C M(c_0) c_0 + C c_1^6 c_0^{10} M(c_0) \\
\leq\ & C c_0 \| \bigtriangledown u_t \|_{L^2} + C c_2 c_1^6 c_0^{11} M(c_0).
\end{aligned}
\tag{2.53}
$$

It follows from the estimates (2.27) and (2.35) that

$$
\begin{aligned}
\|(\bigtriangledown d)^T (\triangle d - f(d))\|_{L^6} &\leq C\|d\|_{H^3}^2 + C\| \bigtriangledown d\|_{L^6} \|d - m\|_{H^2} \|d + m\|_{H^2} \|d\|_{H^2} \\
&\leq C c_2^4 c_1^2 c_0^6 + C c_1^8 c_0^{12} \\
&\leq C c_2^4 c_1^4 c_0^{12}.
\end{aligned}
\tag{2.54}
$$

Taking (2.53) and (2.54) into (2.52), integrating it over time and using the estimate (2.50), we can derive for all $t \in [0, T_3]$,

$$\int_0^t \| \bigtriangledown^2 u \|_{L^6}^2 \mathrm{d}\tau \leq C c_0^2 \int_0^t \| \bigtriangledown u_t \|_{L^2}^2 \mathrm{d}\tau + C c_0^{18} M^2(c_0) \leq C c_0^{18} M^2(c_0).$$

$\square$

It is obvious that Lemma 2.1-Lemma 2.3 imply Theorem 2.1.

2.3 Local existence of a solution to the linear problem (2.1)-(2.3)

Since the estimates obtained in the Lemma 2.1-Lemma 2.3 are uniform for all small δ, we have the following theorem:

Theorem 2.2. *If the initial condition (ρ_0, u_0, d_0) satisfies the regularity (1.8) and the compatibility condition (1.9), then there exists a unique strong solution (ρ, u, d) to the linear equations (2.1)-(2.3) with initial boundary value (1.5)-(1.6) such that*

$$
\begin{aligned}
&\rho \in C([0, T_3]; W^{1,6}), && \rho_t \in C([0, T_3]; L^6), \\
&u \in C([0, T_3]; H_0^1 \cap H^2) \cap L^2(0, T_3; W^{2,6}), && u_t \in L^2(0, T_3; H_0^1), \\
&d \in C([0, T_3]; H^2) \cap L^2(0, T_3; H^3), && d_t \in C([0, T_3]; H_0^1) \cap L^2(0, T_3; H^2), \\
&\sqrt{\rho} u_t \in C([0, T_3]; L^2). &&
\end{aligned}
\tag{2.55}
$$

Moreover, (ρ, u, d) also satisfies the inequalities (2.13)-(2.14), (2.19) and (2.37).

Before proof, we give the following two classical lemmas which are proved in the book [16].

Lemma 2.4. *Let $Y = \{v | v \in L^{\alpha_0}(0, T; X_0), v_t \in L^{\alpha_1}(0, T; X_1)\}$ with norm $|v|_Y = |v|_{L^{\alpha_0}(0,T;X_0)} + |v_t|_{L^{\alpha_1}(0,T;X_1)}$ where $1 < \alpha_0, \alpha_1 < \infty, X_0 \subset X \subset X_1$ are Banach spaces and X_0, X_1 are reflexive. Suppose that the injections $X_0 \hookrightarrow X \hookrightarrow X_1$ are continuous, and the injection from X_0 into X is compact. Then the injection from Y into $L^{\alpha_0}(0, T; X)$ is compact.*

Lemma 2.5. *(An Interpolation Theorem) Let V, H, V' be three Hilbert spaces, each space included in the following one as $V \subset H \equiv H' \subset V'$, V' being the dual of V. If a function u belongs to $L^2(0, T; V)$ and its derivative u_t belongs to $L^2(0, T; V')$, then u is almost everywhere equal to a function continuous form $[0, T]$ into H.*

Proof. We now start to prove Theorem 2.2. From the Theorem 2.1 and the estimates (2.13), (2.14), (2.19) and (2.37), by the compactness Lemma 2.4, there exists (ρ, d, u)

such that

$$(\rho^\delta, d^\delta, u^\delta) \;\rightarrow\; (\rho, d, u) \text{ in } L^2(0, T_3; L^r \times H^2 \times W^{1,q}), \tag{2.56}$$

$$p^\delta \;\overset{*}{\rightharpoonup}\; \widehat{p} \text{ in } L^\infty(0, T_3; W^{1,6}), \quad \text{as } \delta \to 0, \tag{2.57}$$

where $\forall r \in (1, +\infty)$ and $\forall q \in [2, +\infty)$. Hence $p = \widehat{p}, \ a.e..$

Because of the lower semi-continuity of various norms, the estimates (2.13), (2.19) and (2.37) also hold for (ρ, u, d). So for almost every $(t, x)(\in [0, T_3] \times \Omega)$, (ρ, u, d) satisfies the system (2.1)-(2.3) which means (ρ, u, d) is a strong solution to the linear equation (2.1)-(2.3) with initial-boundary conditions (1.5)-(1.6).

The solution (ρ, u, d) is unique: From the Lemma 2.1, ρ is the unique solution of the linear equation (2.1). Using the same method as section 4, we can prove d and u are the unique solution to the linear equations (2.2) and (2.3) respectively.

Finally, we will prove the time continuity of the solution (ρ, u, d). The solution from the Lemma 2.1 is the same as from the approximation (2.56) due to the uniqueness of solution. So we get

$$\rho \in C([0, T]; W^{1,6}). \tag{2.58}$$

From the linear equation (2.1), we easily show

$$\rho_t \in C([0, T_3]; L^6). \tag{2.59}$$

By the interpolation Lemma 2.5, (2.37) can deduce that

$$d_t \in L^2(0, T_3; H^2), \quad d \in L^2(0, T_3; H^3) \Rightarrow d \in C([0, T_3]; H^2). \tag{2.60}$$

Differentiating the linear equation (2.2) with respect to time and space, we get

$$\bigtriangledown d_{tt} + \bigtriangledown(v \cdot \bigtriangledown d)_t = \nu(\bigtriangledown \triangle d_t - \tfrac{1}{\sigma^2} \bigtriangledown [(n + m) \cdot (d - m)n]_t).$$

By the estimate (2.19), we deduce $\bigtriangledown d_{tt} \in L^2(0, T_3; H^{-1})$. Because $\bigtriangledown d_t \in L^2(0, T_3; H^1)$, by the interpolation Lemma 2.5 again, we have

$$d_t \in C([0, T_3]; H_0^1). \tag{2.61}$$

From the linear equation (2.2), we get

$$\triangle d \in C([0, T_3]; H^1). \tag{2.62}$$

By the interpolation Lemma 2.5, (2.37) can deduce that

$$u_t \in L^2(0, T_3; H^1), \quad u \in L^2(0, T_3; W^{2,6}) \Rightarrow u \in C([0, T_3]; H_0^1). \tag{2.63}$$

From the linear equation (2.3) and the estimates (2.13), (2.19) and (2.37), we obtain

$$(\rho u_t, (\rho u_t)_t) \in L^2(0, T_3; H^1) \times L^2(0, T_3; H^{-1}) \Rightarrow \rho u_t \in C([0, T_3]; L^2). \tag{2.64}$$

From the linear equation (2.3) and the elliptic regularity estimate $|\nabla^2 u|_{L^2} \leq C|\triangle u|_{L^2}$, we have

$$u \in C([0, T_3]; H^2). \tag{2.65}$$

So we get time-continuity of (ρ, u, d) from (2.58)-(2.65). $\qquad\square$

3 Iteration and existence in Theorem 1

Set

$$c_1 = Cc_0^{18} M^2(c_0), \quad c_2 = c_1^7, \quad T_3 = \min\{c_2^{-22}, T\}. \tag{3.1}$$

At the beginning, let's choose a initial data of iteration $(u^0(t, x), d^0(t, x))$. $u^0(t, x)$ satisfies the following heat equation

$$\phi_t - \triangle\phi = 0, \text{ with } \phi|_{t=0} = u_0, \ \phi|_{\partial\Omega} = 0,$$

and $d^0(t, x) = d_0(x)$. Because (c_1, c_2, T_3) depends only on c_0, we can choose a $T_*(\in [0, T_3])$ so small that (2.9)-(2.11) hold for u^0 and d^0 with T_* instead of T.

Replacing (v, n) by (u^0, d^0) and using the Theorem 2.2, we can obtain the solution (ρ^1, u^1, d^1) of (2.1)-(2.3) with (1.5)-(1.6), and it satisfies the estimates (2.13), (2.19)

and (2.37). Inductively, for all $k \in N^+$, replacing (v, n) by (u^{k-1}, d^{k-1}) and using the Theorem 2.2, we can obtain a sequence (ρ^k, u^k, d^k) of solution of (2.1)-(2.3) with (1.5)-(1.6), and they satisfy the following estimates with the same (c_0, c_1, c_2, T_*) independent of $k \in N^+$:

$$
\begin{aligned}
& \sup_{0 \leq t \leq T_*} (\|u^k\|_{H_0^1} + \|d^k\|_{H^1} + \|d_t^k\|_{H_0^1} + c_1^{-6}(\|\triangledown^2 u^k\|_{L^2} + \|\triangledown^2 d^k\|_{L^2})) \\
& + \int_0^{T_*} \|\triangledown u_t^k\|_{L^2}^2 + \|u^k\|_{W^{2,6}}^2 + \|\triangledown^2 d_t^k\|_{L^2}^2 + \|d^k\|_{H^3}^2 \mathrm{d}t \\
& \leq C c_0^{22} M^2(c_0)
\end{aligned}
\tag{3.2}
$$

and

$$
\sup_{0 \leq t \leq T_*} (\|\rho^k\|_{W^{1,6}} + \|\rho_t^k\|_{L^6}) \leq C c_2 c_0, \quad \sup_{0 \leq t \leq T_*} \|\sqrt{\rho^k} u_t^k\|_{L^2} \leq C M(c_0) c_0^5,
\tag{3.3}
$$

$$
\sup_{0 \leq t \leq T_*} (\|p^k\|_{W^{1,6}} + \|p_t^k\|_{L^6}) \leq C M(c_0) c_2 c_0, \quad \sup_{0 \leq t \leq T_*} \|\triangledown d^k\|_{H^2} \leq C c_2^2 c_1 c_0^3.
\tag{3.4}
$$

We will show (ρ^k, u^k, d^k) converges to a strong solution to the original nonlinear problem (1.1)-(1.3).

Define

$$
\overline{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \overline{d}^{k+1} = d^{k+1} - d^k, \quad \overline{u}^{k+1} = u^{k+1} - u^k.
\tag{3.5}
$$

Since (ρ^k, u^k, d^k) and $(\rho^{k+1}, u^{k+1}, d^{k+1})$ satisfy the linear equations (2.1)-(2.3), we have

$$
\overline{\rho}_t^{k+1} + \mathrm{div}(\overline{\rho}^{k+1} u^k) + \mathrm{div}(\rho^k \overline{u}^k) = 0,
\tag{3.6}
$$

$$
\begin{aligned}
\overline{d}_t^{k+1} - \nu \triangle \overline{d}^{k+1} = & -\overline{u}^k \cdot \triangledown d^{k+1} - u^{k-1} \cdot \triangledown \overline{d}^{k+1} - \frac{\nu}{\sigma^2}((d^k + m)(d^{k+1} - m))\overline{d}^k \\
& - \frac{\nu}{\sigma^2}((d^k + m) \cdot \overline{d}^{k+1})d^{k-1} - \frac{\nu}{\sigma^2}(\overline{d}^k(d^k - m))d^{k-1},
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
& \rho^{k+1} \overline{u}_t^{k+1} + \rho^{k+1} u^k \cdot \triangledown \overline{u}^{k+1} - \mu \triangle \overline{u}^{k+1} + \triangledown(p^{k+1} - p^k) \\
& = -\lambda(\triangledown \overline{d}^{k+1})^T(\triangle d^{k+1} - f(d^{k+1})) - \lambda(\triangledown d^k)^T(\triangle \overline{d}^{k+1} - (f(d^{k+1}) - f(d^k))) \\
& \quad - \overline{\rho}^{k+1}(u^{k-1} \cdot \triangledown u^k + u_t^k) - \rho^{k+1} \overline{u}^k \cdot \triangledown u^k.
\end{aligned}
\tag{3.8}
$$

Define

$$\Psi^{k+1} = \|\overline{\rho}^{k+1}\|_{L^2}^2 + \|\overline{d}^{k+1}\|_{L^2}^2 + \|\bigtriangledown \overline{d}^{k+1}\|_{L^2}^2 + \|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2. \tag{3.9}$$

Before estimates, we introduce two small positive unfixed constants η and ϵ.

From the first equation (3.6), we can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{\rho}^{k+1}\|_{L^2}^2 \le \mathbb{A}_\eta^k(t)\|\overline{\rho}^{k+1}\|_{L^2}^2 + \eta\|\bigtriangledown \overline{u}^k\|_{L^2}^2, \tag{3.10}$$

where

$$\mathbb{A}_\eta^k(t) = C\|\bigtriangledown u^k(t)\|_{W^{1,6}} + \eta^{-1}C(\|\bigtriangledown \rho^k(t)\|_{L^3}^2 + \|\rho^k(t)\|_{L^\infty}^2).$$

Using the uniform estimates (3.2)-(3.4), we obtain

$$\int_0^t \mathbb{A}_\eta^k(s)\mathrm{d}s \le C + \frac{C}{\eta}t, \qquad \forall t \in [0, T_*]. \tag{3.11}$$

Multiplying (3.7) by $\overline{d}^{k+1}$ and integrating it over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\overline{d}^{k+1}|^2\mathrm{d}x + \int_\Omega |\bigtriangledown \overline{d}^{k+1}|^2\mathrm{d}x$$
$$\le \mathbb{B}_\eta^k(t)\|\overline{d}^{k+1}\|_{L^2}^2 + \eta(\|\bigtriangledown \overline{u}^k\|_{L^2}^2 + \|\bigtriangledown \overline{d}^k\|_{L^2}^2), \tag{3.12}$$

where for all $t \in [0, T_*]$,

$$\mathbb{B}_\eta^k(t) = C\eta^{-1}\|\bigtriangledown d^{k+1}\|_{L^2}^2 + C\|u^{k-1}\|_{L^\infty}^2 + C\eta^{-1}\|d^k + m\|_{L^6}^2\|d^k - m\|_{L^6}^2$$
$$+ C\|d^k + m\|_{L^\infty}\|d^{k-1}\|_{L^\infty} + C\eta^{-1}\|d^{k-1}\|_{L^6}^2\|d^k - m\|_{L^6}^2.$$

The uniform estimates (3.2)-(3.4) implies

$$\int_0^t \mathbb{B}_\eta^k(s)\mathrm{d}s \le C(1 + \frac{1}{\eta})t. \tag{3.13}$$

Multiplying (3.7) by $\triangle\overline{d}^{k+1}$, integrating it over Ω and using the elliptic estimate $\|\bigtriangledown^2 \overline{d}^{k+1}\|_{L^2} \le C\|\triangle\overline{d}^{k+1}\|_{L^2}$, we can deduce by integration by parts

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\bigtriangledown \overline{d}^{k+1}|^2\mathrm{d}x + \int_\Omega |\bigtriangledown^2 \overline{d}^{k+1}|^2\mathrm{d}x \le C\sum_{i=1}^{13} L_i, \tag{3.14}$$

where

$$L_1 = \| \bigtriangledown \overline{u}^k \|_{L^2} \| \bigtriangledown d^{k+1} \|_{L^\infty} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}$$

$$\leq C\eta^{-1} \| \bigtriangledown d^{k+1} \|_{H^2}^2 \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2 + \eta \| \bigtriangledown \overline{u}^k \|_{L^2}^2,$$

$$L_2 = \| \overline{u}^k \|_{L^6} \| \bigtriangledown^2 d^{k+1} \|_{L^3} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}$$

$$\leq C\eta^{-1} \| \bigtriangledown^2 d^{k+1} \|_{H^1}^2 \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2 + \eta \| \bigtriangledown \overline{u}^k \|_{L^2}^2,$$

$$L_3 = \| \bigtriangledown u^{k-1} \|_{W^{1,6}} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2,$$

$$L_4 = \| u^{k-1} \|_{L^\infty} \| \bigtriangledown^2 \overline{d}^{k+1} \|_{L^2} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}$$

$$\leq C\epsilon^{-1} \| u^{k-1} \|_{H^2}^2 \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2 + \epsilon \| \bigtriangledown^2 \overline{d}^{k+1} \|_{L^2}^2,$$

$$L_5 = \| \bigtriangledown d^k \|_{L^3} \| d^{k+1} - m \|_{L^\infty} \| \overline{d}^k \|_{L^6} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}$$

$$\leq C\eta^{-1} \| \bigtriangledown d^k \|_{H^1}^2 \| d^{k+1} - m \|_{H^2}^2 \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2 + \eta \| \bigtriangledown \overline{d}^k \|_{L^2}^2,$$

$$L_6 = \| d^k \|_{L^\infty} \| \bigtriangledown (d^{k+1} - m) \|_{L^3} \| \overline{d}^k \|_{L^6} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}$$

$$\leq C\eta^{-1} \| d^k \|_{H^2}^2 \| \bigtriangledown (d^{k+1} - m) \|_{H^1}^2 \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2 + \eta \| \bigtriangledown \overline{d}^k \|_{L^2}^2,$$

$$L_7 = \| d^k \|_{L^\infty} \| d^{k+1} - m \|_{L^\infty} \| \bigtriangledown \overline{d}^k \|_{L^2} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}$$

$$\leq C\eta^{-1} \| d^k \|_{H^2}^2 \| d^{k+1} - m \|_{H^2}^2 \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2 + \eta \| \bigtriangledown \overline{d}^k \|_{L^2}^2,$$

$$L_8 = \| \bigtriangledown (d^k + m) \|_{L^3} \| \overline{d}^{k+1} \|_{L^6} \| d^{k-1} \|_{L^\infty} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}$$

$$\leq C \| \bigtriangledown (d^k + m) \|_{H^1}^2 \| d^{k-1} \|_{H^2} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2,$$

$$L_9 = \| d^k + m \|_{H^2} \| d^{k-1} \|_{H^2} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2,$$

$$L_{10} = \| d^k + m \|_{L^\infty} \| \overline{d}^{k+1} \|_{L^6} \| \bigtriangledown d^{k-1} \|_{L^3} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}$$

$$\leq C \| d^k + m \|_{H^2}^2 \| \bigtriangledown d^{k-1} \|_{H^1} \| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2,$$

$$L_{11} = \|\triangledown \overline{d}^k\|_{L^2}\|d^k - m\|_{L^\infty}\|d^{k-1}\|_{L^\infty}\|\triangledown \overline{d}^{k+1}\|_{L^2}$$

$$\leq C\eta^{-1}\|d^k - m\|_{H^2}^2\|d^{k-1}\|_{H^2}^2\|\triangledown \overline{d}^{k+1}\|_{L^2}^2 + \eta\|\triangledown \overline{d}^k\|_{L^2}^2,$$

$$L_{12} = \|\overline{d}^k\|_{L^6}\|\triangledown (d^k - m)\|_{L^3}\|d^{k-1}\|_{L^\infty}\|\triangledown \overline{d}^{k+1}\|_{L^2}$$

$$\leq C\eta^{-1}\|\triangledown (d^k - m)\|_{H^1}^2\|d^{k-1}\|_{H^2}^2\|\triangledown \overline{d}^{k+1}\|_{L^2}^2 + \eta\|\triangledown \overline{d}^k\|_{L^2}^2,$$

$$L_{13} = \|\overline{d}^k\|_{L^6}\|d^{k+1} - m\|_{L^\infty}\|\triangledown d^{k-1}\|_{L^3}\|\triangledown \overline{d}^{k+1}\|_{L^2}$$

$$\leq C\eta^{-1}\|d^k - m\|_{H^2}^2\|\triangledown d^{k-1}\|_{H^1}^2\|\triangledown \overline{d}^{k+1}\|_{L^2}^2 + \eta\|\triangledown \overline{d}^k\|_{L^2}^2.$$

Let's ϵ small enough, the inequality (3.14) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\triangledown \overline{d}^{k+1}|^2\mathrm{d}x + \int_\Omega |\triangledown^2 \overline{d}^{k+1}|^2\mathrm{d}x$$

$$\leq C\mathbb{C}_\eta^k(t)|\triangledown \overline{d}^{k+1}|_{L^2}^2 + C\eta(|\triangledown \overline{d}^k|_{L^2}^2 + |\triangledown \overline{u}^k|_{L^2}^2), \tag{3.15}$$

where

$$\mathbb{C}_\eta^k(t) = \eta^{-1}|\triangledown d^{k+1}|_{H^2}^2 + |\triangledown u^{k-1}|_{W^{1,6}} + |u^{k-1}|_{H^2}^2 + \eta^{-1}|\triangledown d^k|_{H^1}^2|d^{k+1} - m|_{H^2}^2$$

$$+\eta^{-1}|d^k|_{H^2}^2|\triangledown (d^{k+1} - m)|_{H^1}^2 + \eta^{-1}|d^k|_{H^2}^2|d^{k+1} - m|_{H^2}^2$$

$$+|\triangledown (d^k + m)|_{H^1}^2|d^{k-1}|_{H^2} + |d^k + m|_{H^2}|d^{k-1}|_{H^2}$$

$$+|d^k + m|_{H^2}^2|\triangledown d^{k-1}|_{H^1} + \eta^{-1}|\triangledown (d^k - m)|_{H^1}^2|d^{k-1}|_{H^2}^2$$

$$+\eta^{-1}|\triangledown (d^k - m)|_{H^1}^2|d^{k-1}|_{H^2}^2 + \eta^{-1}|d^k - m|_{H^2}^2|\triangledown d^{k-1}|_{H^1}^2.$$

The uniform estimates (3.2)-(3.4) implies

$$\int_0^t \mathbb{C}_\eta^k(s)\mathrm{d}s \leq C + (C + \frac{C}{\eta})t, \qquad \forall t \in [0, T_*]. \tag{3.16}$$

Multiplying the movement equation (3.8) by $\overline{u}^{k+1}$ and integrating over Ω, we can

deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \mu\int_\Omega |\bigtriangledown\overline{u}^{k+1}|^2\mathrm{d}x$$

$$\leq \int_\Omega |\overline{\rho}^{k+1}|(|u^{k-1}\cdot\bigtriangledown u^k| + |u_t^k|)|\overline{u}^{k+1}|\mathrm{d}x + \int_\Omega |\rho^{k+1}||\overline{u}^k||\bigtriangledown u^k||\overline{u}^{k+1}|\mathrm{d}x$$

$$+ \lambda\int_\Omega |\bigtriangledown\overline{d}^{k+1}||\triangle d^{k+1} - f(d^{k+1})||\overline{u}^{k+1}|\mathrm{d}x + \lambda\int_\Omega |\bigtriangledown^2 d^k||\bigtriangledown\overline{d}^{k+1}||\overline{u}^{k+1}|\mathrm{d}x$$

$$+ \lambda\int_\Omega |\bigtriangledown d^k||\bigtriangledown\overline{d}^{k+1}||\bigtriangledown\overline{u}^{k+1}|\mathrm{d}x + \frac{1}{\sigma^2}\int_\Omega |\bigtriangledown d^k||d^{k+1} + d^k||\overline{d}^{k+1}||d^{k+1}||\overline{u}^{k+1}|\mathrm{d}x$$

$$+ \frac{1}{\sigma^2}\int_\Omega |\bigtriangledown d^k||d^k + m||d^k - m||\overline{d}^{k+1}||\overline{u}^{k+1}|\mathrm{d}x + \int_\Omega |p^{k+1} - p^k||\bigtriangledown\overline{u}^{k+1}|\mathrm{d}x$$

$$= \sum_{i=1}^8 M_i, \tag{3.17}$$

where we have used

$$f(d^{k+1}) - f(d^k) = \frac{1}{\sigma^2}(\overline{d}^{k+1}\cdot(d^{k+1} + d^k)d^{k+1} + (|d^k|^2 - 1)\overline{d}^{k+1}).$$

Here

$$M_1 \leq \|\overline{\rho}^{k+1}\|_{L^2}\|\overline{u}^{k+1}\|_{L^6}(\|u^{k-1}\|_{L^6}\|\bigtriangledown u^k\|_{L^6} + \|u_t^k\|_{L^3})$$

$$\leq C\|\overline{\rho}^{k+1}\|_{L^2}^2(\|u^{k-1}\|_{H^1}^2\|u^k\|_{W^{2,6}}^2 + \|u_t^k\|_{L^2}\|\bigtriangledown u_t^k\|_{L^2}) + \frac{\mu}{9}\|\bigtriangledown\overline{u}^{k+1}\|_{L^2}^2,$$

$$M_2 \leq \|\sqrt{\rho^{k+1}}\|_{L^6}\|\overline{u}^k\|_{L^6}\|\bigtriangledown u^k\|_{L^6}\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}$$

$$\leq \eta^{-1}\|\sqrt{\rho^{k+1}}\|_{L^6}^2\|u^k\|_{H^2}^2\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2 + \eta\|\bigtriangledown\overline{u}^k\|_{L^2}^2,$$

$$M_3 \leq C\|\bigtriangledown\overline{d}^{k+1}\|_{L^2}\|\triangle d^{k+1} - f(d^{k+1})\|_{L^3}\|\overline{u}^{k+1}\|_{L^6}$$

$$\leq \frac{\mu}{9}\|\bigtriangledown\overline{u}^{k+1}\|_{L^2}^2 + C\|\bigtriangledown\overline{d}^{k+1}\|_{L^2}^2(\|\bigtriangledown^2 d^{k+1}\|_{L^2}\|\bigtriangledown^2 d^{k+1}\|_{L^6}$$

$$+ \|d^{k+1} + m\|_{H^1}^2\|d^{k+1} - m\|_{H^1}^2\|d^{k+1}\|_{H^2}^2),$$

$$M_4 \leq C\|\bigtriangledown^2 d^k\|_{L^3}\|\bigtriangledown \overline{d}^{k+1}\|_{L^2}\|\overline{u}^{k+1}\|_{L^6},$$

$$\leq C\|\bigtriangledown^2 d^k\|_{L^6}\|\bigtriangledown^2 d^k\|_{L^2}\|\bigtriangledown \overline{d}^{k+1}\|_{L^2}^2 + \frac{\mu}{9}\|\bigtriangledown \overline{u}^{k+1}\|_{L^2}^2$$

$$M_5 \leq C\|\bigtriangledown d^k\|_{L^\infty}\|\bigtriangledown \overline{d}^{k+1}\|_{L^2}\|\bigtriangledown \overline{u}^{k+1}\|_{L^2}$$

$$\leq C\|\bigtriangledown d^k\|_{W^{1,6}}^2\|\bigtriangledown \overline{d}^{k+1}\|_{L^2}^2 + \frac{\mu}{9}\|\bigtriangledown \overline{u}^{k+1}\|_{L^2}^2,$$

$$M_6 \leq C\|\bigtriangledown d^k\|_{L^3}\|d^{k+1} + d^k\|_{L^6}\|\overline{d}^{k+1}\|_{L^6}\|d^{k+1}\|_{L^6}\|\overline{u}^{k+1}\|_{L^6}$$

$$\leq C\|\bigtriangledown d^k\|_{L^2}\|\bigtriangledown d^k\|_{H^1}\|d^{k+1} + d^k\|_{H^1}^2\|d^{k+1}\|_{H^1}^2\|\bigtriangledown \overline{d}^{k+1}\|_{L^2}^2$$

$$+ \frac{\mu}{9}\|\bigtriangledown \overline{u}^{k+1}\|_{L^2}^2,$$

$$M_7 \leq C\|\bigtriangledown d^k\|_{L^3}\|d^k + m\|_{L^6}\|d^k - m\|_{L^6}\|\overline{d}^{k+1}\|_{L^6}\|\overline{u}^{k+1}\|_{L^6}$$

$$\leq C\|\bigtriangledown d^k\|_{L^2}\|\bigtriangledown d^k\|_{H^1}\|d^k + m\|_{H^1}^2\|d^k - m\|_{H^1}^2\|\bigtriangledown \overline{d}^{k+1}\|_{L^2}^2$$

$$+ \frac{\mu}{9}\|\bigtriangledown \overline{u}^{k+1}\|_{L^2}^2,$$

$$M_8 \leq \|p^{k+1} - p^k\|_{L^2}\|\bigtriangledown \overline{u}^{k+1}\|_{L^2}$$

$$\leq CM^2(c_0)\|\overline{\rho}^{k+1}\|_{L^2}^2 + \frac{\mu}{9}\|\bigtriangledown \overline{u}^{k+1}\|_{L^2}^2.$$

Taking $M_1 - M_8$ into (3.17), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \int_\Omega |\bigtriangledown \overline{u}^{k+1}|^2\mathrm{d}x$$

$$\leq C\mathbb{D}_\eta^k(t)(\|\overline{\rho}^{k+1}\|_{L^2}^2 + \|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2 + \|\bigtriangledown \overline{d}^{k+1}\|_{L^2}^2) + C\eta\|\bigtriangledown \overline{u}^k\|_{L^2}^2, \qquad (3.18)$$

where

$$\mathbb{D}_\eta^k(t) = \|u^{k-1}\|_{H^1}^2\|u^k\|_{W^{2,6}}^2 + \|u_t^k\|_{L^2}^2\|\bigtriangledown u_t^k\|_{L^2} + \eta^{-1}\|\sqrt{\rho^{k+1}}\|_{L^6}^2\|u^k\|_{H^2}^2$$

$$+ \|\bigtriangledown^2 d^{k+1}\|_{L^2}\|\bigtriangledown^2 d^{k+1}\|_{L^6} + \|d^{k+1} + m\|_{H^1}^2\|d^{k+1} - m\|_{H^1}^2\|d^{k+1}\|_{H^2}^2$$

$$+ \|\bigtriangledown^2 d^k\|_{L^6}\|\bigtriangledown^2 d^k\|_{L^2} + \|\bigtriangledown d^k\|_{W^{1,6}}^2 + \|\bigtriangledown d^k\|_{L^2}\|\bigtriangledown d^k\|_{H^1}$$

$$\cdot (\|d^{k+1} + d^k\|_{H^1}^2\|d^{k+1}\|_{H^1}^2 + \|d^k + m\|_{H^1}^2\|d^k - m\|_{H^1}^2) + M^2(c_0).$$

From the uniform estimates (3.2)-(3.4), we have

$$\int_0^t \mathbb{D}_\eta^k(s)\mathrm{d}s \leq C + (C + \frac{C}{\eta})t, \qquad \forall t \in [0, T_*]. \qquad (3.19)$$

Summing (3.10), (3.12), (3.15) and (3.18), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \Psi^{k+1} &+ (\| \nabla \overline{d}^{k+1} \|_{L^2}^2 + \| \nabla^2 \overline{d}^{k+1} \|_{L^2}^2 + \| \nabla \overline{u}^{k+1} \|_{L^2}^2) \\
&\leq C \mathbb{E}_\eta^k(t) \Psi^{k+1} + C\eta(\| \nabla \overline{u}^k \|_{L^2}^2 + \| \nabla \overline{d}^k \|_{L^2}^2),
\end{aligned}
\tag{3.20}
$$

where

$$
\mathbb{E}_\eta^k(t) = \mathbb{A}_\eta^k(t) + \mathbb{B}_\eta^k(t) + \mathbb{C}_\eta^k(t) + \mathbb{D}_\eta^k(t).
$$

Using the uniform estimates (3.2)-(3.4), we obtain again

$$
\int_0^t \mathbb{E}_\eta^k(s)\mathrm{d}s \leq C + C(1 + \eta^{-1})t, \qquad \forall t \in [0, T_*].
\tag{3.21}
$$

Applying Gronwall's inequality to (3.20), we can deduce

$$
\begin{aligned}
\Psi^{k+1}(t) &+ \int_0^t (\| \nabla \overline{d}^{k+1} \|_{L^2}^2 + \| \nabla^2 \overline{d}^{k+1} \|_{L^2}^2 + \| \nabla \overline{u}^{k+1} \|_{L^2}^2)\mathrm{d}s \\
&\leq C\eta \int_0^t (\| \nabla \overline{u}^k \|_{L^2}^2 + \| \nabla \overline{d}^k \|_{L^2}^2)\mathrm{d}\tau \exp(C + C(1 + \eta^{-1})t).
\end{aligned}
\tag{3.22}
$$

Hence choose small constants η, $T^*(< T_*)$, so that

$$
C\eta \exp(C + C(1 + \eta^{-1})t) \leq \frac{1}{2}, \qquad \forall t \in [0, T^*].
\tag{3.23}
$$

We easily deduce that

$$
\begin{aligned}
\sum_{k=1}^\infty &\sup_{0 \leq t \leq T^*} \Psi^{k+1}(t) + \sum_{k=1}^\infty \int_0^{T^*} (\| \nabla \overline{d}^{k+1} \|_{L^2}^2 + \| \nabla^2 \overline{d}^{k+1} \|_{L^2}^2 + \| \nabla \overline{u}^{k+1} \|_{L^2}^2)\mathrm{d}s \\
&\leq C < \infty.
\end{aligned}
\tag{3.24}
$$

(3.24) implies that the full sequence (ρ^k, d^k, u^k) converges to a limit (ρ, d, u) in the following strong sense

$$
\begin{aligned}
&\rho^k \to \rho \text{ in } L^\infty(0, T^*; L^2), \qquad\qquad\quad u^k \to u \text{ in } L^2(0, T^*; H_0^1), \\
&d^k \to d \text{ in } L^\infty(0, T^*; H^1) \cap L^2(0, T^*; H^2).
\end{aligned}
\tag{3.25}
$$

Hence the problem (1.1)-(1.3) with initial boundary data (1.5)-(1.6) has a weak solution (ρ, u, d). Furthermore using the estimates (3.2)-(3.4), we obtain that a subsequence of

(ρ^k, u^k, d^k) converges to (ρ, u, d) in an obvious weak or weak* sense. Due to the lower semi-continuity of various norms, from (3.2)-(3.4), (ρ, u, d) also satisfies the following regularity estimates

$$
\begin{aligned}
\sup_{0 \leq t \leq T^*} & (\|\rho\|_{W^{1,6}} + \|\rho_t\|_{L^6} + \|u\|_{H_0^1} + \|p\|_{W^{1,6}} + \|p_t\|_{L^6} + \|d\|_{H^1} + \|d_t\|_{H_0^1} \\
& + \| \nabla^2 u\|_{L^2} + \| \nabla^2 d\|_{L^2} + \| \nabla d\|_{H^2}) \\
+ \int_0^{T^*} & \|\sqrt{\rho}u_t\|^2 + \| \nabla u_t\|_{L^2}^2 + \|u\|_{W^{2,6}}^2 + \| \nabla^2 d_t\|_{L^2}^2 + \|d\|_{H^3}^2 \mathrm{d}t \\
\leq C. &
\end{aligned} \tag{3.26}
$$

Hence (ρ, u, d) is also a strong solution to the problem (1.1)-(1.3).

4   Uniqueness and continuity in Theorem 1

In this section, we will use energy method to prove the uniqueness and continuity in Theorem 1. For simplicity, we introduce some notations

$$
\overline{\rho} = \rho - \widetilde{\rho}, \ \ \overline{u} = u - \widetilde{u}, \ \ \overline{d} = d - \widetilde{d}.
$$

Define

$$
\Psi(t) = \|\overline{\rho}\|_{L^2}^2 + \|\overline{d}\|_{L^2}^2 + \| \nabla \overline{d}\|_{L^2}^2 + \|\sqrt{\rho}\overline{u}\|_{L^2}^2.
$$

Using the similar process in the section 3, we can obtain the following estimate (see (3.22))

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Psi + (\| \nabla \overline{d}\|_{L^2}^2 + \| \nabla^2 \overline{d}\|_{L^2}^2 + \| \nabla \overline{u}\|_{L^2}^2) \leq C\mathbb{F}(t)\Psi, \tag{4.1}
$$

where $\mathbb{F}(t) \in L^1(0, T^*)$.

Applying the Gronwall's inequality to (4.1), we get for all $t \in [0, T^*]$,

$$
\Psi(t) + \int_0^t (\| \nabla^2 \overline{d}\|_{L^2}^2 + \| \nabla \overline{u}\|_{L^2}^2)\mathrm{d}\tau \leq C\Psi(0), \tag{4.2}
$$

which implies the uniqueness, and for all $t \in [0, T^*]$, we have

$$(\|\overline{\rho}\|_{L^2}^2 + \|\overline{d}\|_{H^1}^2 + \|\sqrt{\rho}\overline{u}\|_{L^2}^2)(t) + \int_0^t (\|\nabla^2 \overline{d}\|_{L^2}^2 + \|\nabla \overline{u}\|_{L^2}^2)\mathrm{d}\tau \to 0 \tag{4.3}$$

as $(\widetilde{\rho}_0, \widetilde{u}_0, \widetilde{d}_0) \to (\rho_0, u_0, d_0)$ in $W^{1,6} \times H^2 \times H^3$.

Because d and $\widetilde{d}$ satisfy (1.3), we obtain, similar to (3.7),

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\triangle \overline{d}|^2 \mathrm{d}x + \int_\Omega |\nabla \overline{d_t}|^2 \mathrm{d}x$$
$$\leq C(\|\nabla \overline{u}\|_{L^2}^2 + \|\nabla \overline{d}\|_{L^2}^2 \|\nabla \widetilde{u}\|_{L^6}^2 + \|\nabla \overline{d}\|_{L^2}^2) + \|\widetilde{u}\|_{L^\infty}^2 \|\triangle \overline{d}\|_{L^2}^2.$$

Applying Gronwall's inequality to the above inequality, and using the inequality (4.3) and the elliptic estimate $\|\overline{d}\|_{H^2} \leq C\|\triangle \overline{d}\|_{L^2}$, we have, for all $t \in [0, T^*]$,

$$\|\overline{d}\|_{H^2}(t) + \int_0^t \|\nabla \overline{d_t}\|_{L^2}^2 \mathrm{d}\tau \to 0 \tag{4.4}$$

as $(\widetilde{\rho}_0, \widetilde{u}_0, \widetilde{d}_0) \to (\rho_0, u_0, d_0)$ in $W^{1,6} \times H^2 \times H^3$.

Similarly from (3.8), using Gronwall's inequality, (3.26) and the convergence (4.3)-(4.4), we obtain, for all $t \in [0, T^*]$,

$$\|\overline{u}\|_{H^1}(t) + \int_0^{T^*} \|\sqrt{\rho}\overline{u_t}\|_{L^2}^2 \mathrm{d}t \to 0 \tag{4.5}$$

as $(\widetilde{\rho}_0, \widetilde{u}_0, \widetilde{d}_0) \to (\rho_0, u_0, d_0)$ in $W^{1,6} \times H^2 \times H^3$.

Multiplying the difference between the continuity equations by $6\overline{\rho}^5$, integrating over $(0, t) \times \Omega$ and then using Gronwall's inequality, it follows from the estimate (3.26) and the convergence (4.5) that for all $t \in [0, T^*]$,

$$\|\rho - \widetilde{\rho}\|_{L^6}(t) \to 0. \tag{4.6}$$

From the equations (1.2) and (1.3), by a simple discussion, we can obtain for all $t \in [0, T^*]$,

$$\|\overline{d}\|_{L^2}(t), \ \|\overline{d}\|_{L^2(0,T^*;H^3)}, \ \|\overline{u}\|_{L^2(0,T^*;H^2)} \ \to 0 \tag{4.7}$$

as $(\widetilde{\rho}_0, \widetilde{u}_0, \widetilde{d}_0) \to (\rho_0, u_0, d_0)$ in $W^{1,6} \times H^2 \times H^3$.

In conclusion, (4.3)-(4.7) complete the proof of the continuity in Theorem 1.

5   Proof of Theorem 2

Suppose that there are two positive constants $\theta(< 1)$ and $\widetilde{C}$ such that

$$\max\{\|\rho_0\|_{W^{1,6}}, \|u_0\|_{H^2}, \|d_0 - m\|_{H^3}, \|g\|_{L^2}^2\} < \theta, \tag{5.1}$$

$$\sup_{0 \le t \le T}(\|v\|_{H^2} + \|n - m\|_{H^2} + \|n_t\|_{H_0^1}) + \int_0^T \|\nabla v_t\|_{L^2}^2 + \|v\|_{W^{2,6}}^2$$
$$+ \|\nabla^2 n_t\|_{L^2}^2 + \|n\|_{H^3}^2 \mathrm{d}t < \widetilde{C}. \tag{5.2}$$

In this section, we assume the genuine constant C, maybe depending on the constant $M(1)$ which is defined by (2.18).

By Lemma 2.1, there exists a small $\theta_1(< 1)$ so that $\forall t \in [0, T]$, $\forall \theta \in (0, \theta_1]$,

$$\|\rho\|_{W^{1,6}} \le C\theta^{\frac{1}{2}}, \quad \|\rho_t\|_{L^6} \le C\theta^{\frac{1}{3}}, \quad \|p\|_{W^{1,6}} \le C\theta^{\frac{1}{2}}, \quad \|p_t\|_{L^6} \le C\theta^{\frac{1}{3}}, \tag{5.3}$$

From Lemma 2.2, we can find a small $\theta_2(< 1)$ so that $\forall \theta \in (0, \theta_2]$,

$$\|d_t\|_{H^1}^2(t), \ \|d - m\|_{H^2}^2(t), \ \int_0^t \|d - m\|_{H^3}^2 \mathrm{d}\tau \ \le C\theta^{\frac{1}{2}}, \ \forall t \in [0, T]. \tag{5.4}$$

By Lemma 2.3, a small $\theta_3(\le \min\{\theta_1, \theta_2\})$ also can be found so that $\forall \theta \in (0, \theta_3]$,

$$\|u\|_{H^2}^2(t), \ \|\sqrt{\rho}u_t\|_{L^2}^2, \ \int_0^t \|u_t\|_{H^1}^2 \mathrm{d}\tau, \ \int_0^t \|u\|_{W^{2,6}}^2 \mathrm{d}\tau \ \le C\theta^{\frac{1}{6}}, \ \forall t \in [0, T]. \tag{5.5}$$

Thanks to the estimates (5.3)-(5.5), using the Theorem 2.2, we can obtain the global strong solution of the linear system (2.1)-(2.3) with initial boundary value (1.5) and (1.6) provided

$$\max\{\|\rho_0\|_{W^{1,6}}, \|u_0\|_{H^2}, \|d_0 - m\|_{H^3}, \|g\|_{L^2}^2\} \le \theta,$$

where $\forall \theta \in (0, \theta_3]$.

Now let's talk about the iteration. First, we notice that if θ_3 is taken so small that $C\theta_3 \le \widetilde{C}$, then the process of iteration can be continued for the same θ_3. Next we will pay attention to the convergence of the iteration.

As the same process in the section 3, using the estimates (5.3)-(5.5), we can obtain, like (3.22),

$$\Psi^{k+1}(t) + \int_0^t (\| \bigtriangledown \overline{d}^{k+1} \|_{L^2}^2 + \| \bigtriangledown^2 \overline{d}^{k+1} \|_{L^2}^2 + \| \bigtriangledown \overline{u}^{k+1} \|_{L^2}^2) \mathrm{d}s$$

$$\leq C\eta \exp(C(t + \theta^{\frac{1}{6}}t + \theta^{\frac{1}{6}} + \eta^{-1}\theta^{\frac{1}{2}}t + \eta^{-1}\theta^{\frac{1}{2}})) \int_0^t (\| \bigtriangledown \overline{u}^k \|_{L^2}^2 + \| \bigtriangledown \overline{d}^k \|_{L^2}^2) \mathrm{d}\tau, \quad (5.6)$$

where

$$\Psi^{k+1} = \|\overline{\rho}^{k+1}\|_{L^2}^2 + \|\overline{d}^{k+1}\|_{L^2}^2 + \| \bigtriangledown \overline{d}^{k+1}\|_{L^2}^2 + \|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2.$$

Hence choose small constants $\eta, \ \theta_0$, so that $\forall \theta \in (0, \theta_0]$ and $\forall t \in [0, T]$,

$$C\eta \exp(C(t + \theta^{\frac{1}{6}}t + \theta^{\frac{1}{6}} + \eta^{-1}\theta^{\frac{1}{2}}t + \eta^{-1}\theta^{\frac{1}{2}})) \leq \frac{1}{2}, \quad (5.7)$$

We easily deduce that $\forall t \in [0, T]$,

$$\sum_{k=1}^{\infty} \sup_{0 \leq t \leq T} \Psi^{k+1}(t) + \sum_{k=1}^{\infty} \int_0^T (\| \bigtriangledown \overline{d}^{k+1}\|_{L^2}^2 + \| \bigtriangledown^2 \overline{d}^{k+1}\|_{L^2}^2 + \| \bigtriangledown \overline{u}^{k+1}\|_{L^2}^2) \mathrm{d}s$$

$$\leq C \leq \infty. \quad (5.8)$$

So we complete the proof of Theorem 2.

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